ColoMATYC 2020

Building a Differential Equations Course on Collaborative Overleaf Projects

Kenneth M Monks

Front Range Community College - Boulder County Campus
Department of Mathematics
kenneth.monks@frontrange.edu

Conundrum

1. At this point, the educational benefits of active learning are about as well-researched as the health risks of smoking cigarettes. I really need to incorporate active learning into my class.
2. I have chronically poor organizational skills and short-term memory. *"Wait, was this class prep? Or an in-class activity? Or a homework? Which column do I put the grade in?"*

My short-term memory was about to get far worse

"Where did I leave my toothbrush?"

Solution

I needed just *one* type of learning object for my Differential Equations/Linear Algebra course that could serve *every* purpose: class prep, in-class activity, homework, references, etc.

"Why not just have everything in Overleaf documents? I can't lose those."

Overleaf is a free cloud-based shared editing platform for LaTeX documents.

Cut the Course into Eight Parts

1. Solving first-order DEs (separation of variables, integrating factors, substitution methods, and power series)
2. Existence and Uniqueness Theorem for Solultions to Initial Value Problems
3. Higher order linear DEs and the methods of characteristic polynomial/undetermined coefficients
4. Proof by Induction
5. Laplace Transforms
6. Vector Spaces, Bases, and Determinants
7. Linear Transformations and Eigenstuff
8. Solving Linear Systems of DEs and Nonlinear Approximation via Jacobians

Write One Overleaf Project per Part

Each of the eight Overleaf projects contains the following:

- **Notes** on the topic (definitions, statements of theorems, some motivating prose)
- Fully solved **Examples**, worked out by me
- **Exercises** for the students with empty solution boxes
- **Comment** boxes under all Examples and Exercises
- A section linking to recommended **Resources**: other notes, texts, videos, MIT OpenCourseware, etc

Inside an Overleaf Project: Definitions and Theorems

Higher-Order Linear Homogeneous Differential Equations

Definition 0.0.0.1. Linear Differential Equation

Let $n \in \mathbb{N}$. An order n linear differential equation is an equation of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = g(x)$$

for some functions $p_1(x), p_2(x), \ldots, p_n(x), g(x)$. In that form, the right-hand side $g(x)$ is called the *forcing term*. If $g(x) = 0$, the equation is called *homogeneous*.

Theorem 0.0.0.2. Structure of Solutions to Linear Differential Equations

Given the differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = g(x),$$

construct the *associated homogeneous equation*, also sometimes called the *complimentary equation*,

$$y_h^{(n)} + p_1(x)y_h^{(n-1)} + p_2(x)y_h^{(n-2)} + \cdots + p_n(x)y_h = 0.$$

That is, simply set the right-hand side equal to zero. If y_h represents the general solution to the associated homogeneous equation, and y_p is any particular solution to the original (possibly nonhomogeneous) differential equation, then the general solution is of the form

$$y_h + y_p.$$

Thus, we see that there are three steps to solving a linear DE:

Inside an Overleaf Project: Notes

- First, find the **homogeneous solution** y_h.
- Second, find a **particular solution** y_p.
- Lastly, add those together to obtain the **general solution** $y_h + y_p$.

In general it is too hard to solve unless the coefficients are constant functions. In this case though, we have nice algorithms for solving.

In the next session, we focus on how to find the homogeneous solution. In the following, we will discuss how to find the particular solution.

0.0.1 Finding a Homogeneous Solution

It turns out to find the homogeneous solution, we only need the following observation:

If you can take a function, take a bunch of its derivatives, add them together, and zero comes out the other side, it was probably e to the something.

Here's the idea. Say you plug in $y(x) = \ln(x)$; there is no way that will solve a linear differential equation of any order, because the first derivative will be $y = 1/x$. How on earth will a log cancel a rational function when added together? Exactly. However when you differentiate an exponential, you get another exponential. This gives some hope of them all cancelling out to zero and solving the homogeneous DE.

Moral to the story is that finding the homogeneous solution is quite easy! Simply **plug in the educated guess**

$$y(x) = e^{\lambda x}$$

and then solve for λ. Now the question remains, what do we do with these different λ values to build the general solution? This is answered by the theorem below.

Theorem 0.0.1.1. Superposition Principle

If functions $f(x)$ and $g(x)$ are both solutions to a linear homogeneous DE, then so is

$$C_1 f(x) + C_2 g(x)$$

for all real numbers C_1 and C_2.

Inside an Overleaf Project: Worked Out Examples...

Exercise 0.0.1.7. Repeated Roots!

Find the general solution to the differential equation

$$y^{(4)} - 11y^{(3)} + 45y^{(2)} - 81y^{(1)} + 54y = 0.$$

Solution Author: Kenneth M Monks

Since it is a linear homogeneous differential equation, we begin by plugging in our educated guess of $y(x) = e^{\lambda x}$. This produces the characteristic polynomial

$$\lambda^4 - 11\lambda^3 + 45\lambda^2 - 81\lambda + 54.$$

To find the roots of this polynomial, there is no clear path via factor by grouping. Instead, we simply try some plausible guesses for roots based on the Rational Root Theorem. In particular, RRT guarantees that the only possible rational roots of that polynomial are on the list below:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm 27, \pm 54.$$

We now simply go through the list from left to right until we find a root.

- **Guess $\lambda = 1$.** Plugging in $\lambda = 1$ produces

  $$1^4 - 11 \cdot 1^3 + 45 \cdot 1^2 - 81 \cdot 1 + 54 = 1 - 11 + 45 - 81 + 54 = 8,$$

  so 1 is not a root.

- **Guess $\lambda = -1$.** Plugging in $\lambda = -1$ produces

  $$(-1)^4 - 11 \cdot (-1)^3 + 45 \cdot (-1)^2 - 81 \cdot (-1) + 54 = 1 + 11 + 45 + 81 + 54$$

Inside an Overleaf Project: ...with comment boxes!

$$\begin{aligned}\lambda^3 - 9\lambda^2 + 27\lambda - 27 &= \left(\lambda^3 - 27\right) + \left(-9\lambda^2 + 27\lambda\right)\\ &= (\lambda - 3)\left(\lambda^2 + 3\lambda + 9\right) - 9\lambda(\lambda - 3)\\ &= (\lambda - 3)\left(\lambda^2 + 3\lambda + 9 - 9\lambda\right)\\ &= (\lambda - 3)\left(\lambda^2 - 6\lambda + 9\right)\\ &= (\lambda - 3)(\lambda - 3)^2\\ &= (\lambda - 3)^3.\end{aligned}$$

Thus our characteristic polynomial factors as

$$\lambda^4 - 11\lambda^3 + 45\lambda^2 - 81\lambda + 54 = (\lambda - 3)^3(\lambda - 2).$$

The root of $\lambda = 2$ has multiplicity 1, so it tells us that

$$e^{2x}$$

is a solution to the differential equations. However, the root $\lambda = 3$ has multiplicity 3, so it tells us that

$$e^{3x}, xe^{3x}, x^2e^{3x}$$

are all solutions of our DE, using Theorem 6 with $n = 3$ and $r = 3$. To build the general solution, we now apply Theorem 1, producing

$$y(x) = C_1e^{2x} + C_2e^{3x} + C_3xe^{3x} + C_4x^2e^{3x}$$

for real numbers C_1, C_2, C_3, C_4.

Comments on the Above

Inside an Overleaf Project: Exercises

Exercise 0.0.2.7. ☕☕☕

Find the general solution to the differential equation

$$y''' - 3y'' - 4y' + 12 = 0.$$

Verify your answer by plugging back into the DE!

Solution Author: Insert Name Here!

Comments on the Above

Inside an Overleaf Project: Resources

- **Herbert Gross MIT Opencourseware.** Video lectures from the original distance-learning course, developed by Herbert Gross of MIT/Bunker Hill Community College. They provide a beautiful mix of high level conceptual framework along with some really concrete examples.

  `https://ocw.mit.edu/resources/res-18-008-calculus-revisited-complex-variables-differential-equations-and-linear-algebra-fall-2011/part-ii/lecture-1-the-concept-of-a-general-solution/`

- **PatrickJMT's Youtube Channel.** Very low tech but mathematically high quality youtube channel. No bells and whistles, just good exposition of short examples that he works out.
  - **Introduction to DEs and Solutions.** `https://www.youtube.com/watch?v=6n-TgcDTAPw&list=PLD4B0062CA82D73FB&index=10`
  - **Separable.** `https://www.youtube.com/watch?v=uS_5bmRUYEI&list=PLD4B0062CA82D73FB&index=9`
  - **Power Series Solutions.** `https://www.youtube.com/watch?v=6csP7dw0XTY&index=25&list=PLD4B0062CA82D73FB`
  - **Integrating Factors.** `https://www.youtube.com/watch?v=RnYzatmp-_s`
  - **Bernoulli Substitution.** `https://www.youtube.com/watch?v=hNCE3AxbWj0&index=34&list=PLD4B0062CA82D73FB`
  - **Homogeneous of Degree n.** `https://www.youtube.com/watch?v=vEtEAYi2cIA&list=PLD4B0062CA82D73FB&index=15`

Course Structure

I tell the students that they want to read the entire project and work all problems (just on paper) to be prepared for the tests. Together, in the Overleaf document, we are building the answer key to the project.

What happens in each project

- **Initial Draft.** Each student must draft a solution to 1 problem. It is fine if they get stuck! They can write partial progress.
- **Peer Review.** Each student writes a peer review for another student draft. They are also welcome to comment or ask questions more informally on any other problems.
- **Instructor Review.** I go in and leave feedback after the peer reviews finish.
- **Revise and Complete.** The students are responsible for responding to the feedback and incorporating it as appropriate.

Because the course involves so much writing in LaTeX, I do have *one* assignment outside of the eight projects: a LaTeX Quick Start Guide that also establishes a set of writing style guidelines for our course.

Student Work

Exercise 0.0.0.27. An IVP

Consider the IVP given below.

Differential Equation	$y'' - 6y' + 9y = e^{3x} + e^{x}$
Initial Condition	$y(0) = -1$ and $y'(0) = 2$

- What did we get when we solved this back in the characteristic polynomial/MUC assignment?
- Use Laplace transforms to solve it again! Confirm your answers match.

Proof 27. Author: [redacted]

Begin by applying the Laplace Transform to both sides of the equation. The left-hand side of the equation becomes

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 9\mathcal{L}(y)$$

Using the table of known Laplace transforms, these quantities become

$$s^2F(s) - sy(0) - y'(0) - 6(sF(s) - y(0)) + 9F(s).$$

Plugging in the initial conditions helps in some further simplification. The expression simplifies to understand where these come from. Application of Inverse Laplace leads to the equation

$$y(x) = \frac{1}{2}e^{3t}t^2 + \frac{1}{4}e^t - \frac{1}{4}e^{3t} + \frac{1}{2}e^{3t}t - e^{3t} - 3te^{3t} + 8te^{3t}.$$

Which finally simplifies to

$$y(x) = \frac{1}{2}e^{3t}t^2 + \frac{1}{4}e^t - \frac{5}{4}e^{3t} + \frac{11}{2}e^{3t}t.$$

The answer that [redacted] came up with in the M.U.C. project was

$$y(x) = -e^{3x} + 5xe^{3x} + \frac{1}{2}x^2e^{3x} + \frac{1}{4}e^x.$$

Which makes me wonder...

Comments on Proof 27.

- **Feedback from Ken:** [redacted]! I fear something went wrong kind of early. In particular, I like the step
$$s^2F(s) - sy(0) - y'(0) - 6(sF(s) - y(0)) + 9F(s)$$
but then when you simplify it I think something went really funny. The $F(s)$ should be multiplied by $s^2 - 6s + 9$, no? Not just $s^2 + 3$ as you have written. That will then affect things a lot throughout the problem.
- **Response from** [redacted]: Yuuuuuup. I sure did :P I'll get to redoing this now. At least my heart was in the right place.
- **Update from** [redacted]: Should be good to go, Ken.

Student Work

- **Peer Review from** [redacted] I think this was an excellent proof and very well written. I like how you form yo[redacted]ntences. I also noticed it is more succinct than my writing; I tend to write every step out whereas you skipped the basic Algebra as an exercise for the reader. I think I like your way better at this level. I was able to focus on the Laplacian logic and just stopped to check the results as an aside.

  A few small things I would change in the writeup:

  1. You missed the part about confirming it matched the MUC solution.
  2. Just at the beginning, where you transform the left-hand side, I see that as the application of Exercise 0.0.0.2, the Laplace of a first derivative, not the table of known LaPlace transforms, which I see as straight functions. I also thought it might be worth mentioning that you applied the derivative transform twice on the first term.
  3. And further down, where you transform the right-hand side, I do see that as using the table of known LaPlace transforms.
  4. I would change 'Now move to isolating...' into 'Now move to isolate...'.
  5. I would not assume the reader knows which term needs PFD, or that they even know what PFD is. I would also have said we need distinct denominators so that we can fit individual terms to the table of known inverse Laplacians. But that is really a Ken question: Can we assume all of the readers are classmates or do we need to write for a general audience, as if this were a textbook? I've been doing the latter.
  6. Finally, I would have shown the actual PFD work, especially since we are in the process of combining Linear Algebra with Diff Eq. This PFD would be a good lead-in to that work.

- **Reply from** [redacted]: Hi [redacted]. Thanks for the thorough peer review. I'll respond to each of your 6 find[redacted]gs below [redacted] the same format that they were brought up.

  1. I posted the answer that [redacted] found in the M.U.C. assignment. Scary thing was...they didn't match! I'll work out the M.U.C. version and see who needs to correct it.
  2. I think you're talking about the Laplace transform of y''? For reasons I don't know of, my brain does not like applying the "first derivative transform twice" method. For any derivative transform greater than one, I always use the n-th derivative transform

$$\mathcal{L}(f^{(n)}(t) = s^n F(s) - s^{s-1} f(0) - ... - f^{n-1} f(0).$$

Results: How did it affect me?

- I can leave far far better feedback in far less time than paper/pen ever could.
- No more wasting time decoding student handwriting.
- Less stacks of diseased homeworks brought home, where my child lives.

Results: How did it affect the students?

- They don't have to read my handwriting anymore!
- Benefits of active learning are now present even in their homework!
- They leave the course knowing how to create a well-written technical document, not only using LaTeX but also style, grammar, etc. Easiest way *ever* to learn LaTeX or get into programming for the first time.
- Zero-cost course materials. (18% of our students are or have been homeless) OER isn't limited to the idea of a free textbook.

At least anecdotally, it has been a *huge* success across a very diverse group of students.

"I just wanted to reach out again to thank you for all of your help in my academic journey. I'm now tutoring Diffeq/Linear algebra for CU Denver and it's been great! If someone had told me that in a year from when I took it I would be tutoring this class I wouldn't have ever believed them lol. " -19 year old college student who came into the course with math anxiety

Results: Student Testimonials

"I'm writing simply to thank you again for getting me started on Overleaf. I'm now starting to use rmarkdown, with is written directly in RStudio, and leverages LaTeX syntax for math (note below printscreen). There is no preview as with Overleaf though, so had I not used it so much to date, I guarantee this would be brutal."

- Mid 40s woman working in Business Analytics at Spectra Logic, Pursuing Colorado State University's Master's of Applied Stats

Results: Student Testimonials

"I'm now at CU Boulder pursuing a Physics major and, largely because of you, an Applied Math minor (maybe double major)...The idea of teaching myself LaTex now with my course load is terrifying, and the way we learned LaTex in your class was so natural yet I had no idea how grateful I would be that you incorporated it into our class." -Late 20s, tough chef-y dude

Ok hippie community college prof, enough thoughts and feelings. Get a haircut..

What has it done to pass rates?

- Traditional textbook/homeworks differential equations, FA16 - FA18. Pass rate: 70.1%
- Post-Overleaf Rebuild, FA19 and SP19. Pass rate: 70.5%

Sharing and Thanks!

All materials mentioned in this talk (the eight projects plus the LaTeX Quick Start Guide) are available from the author upon request: kenneth.monks@frontrange.edu.